Sections 4.1-4.2

Sec 4.1: Review of Basic Calculus of Matrix Functions

Definition: A **Matrix Function** is a matrix whose entries are functions. In this class we will consider matrices whose entries are real valued functions of a real number t.

Ex1. Consider the matrix

$$M(t) = \begin{bmatrix} t^2 - t & 3 \\ t-1 & t-2 \end{bmatrix}$$

(a) Compute $M'(t)$ and $\int M(t)\,dt$.

$$M'(t) = \begin{bmatrix} 2t-1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \int M(t)\,dt = \begin{bmatrix} \int (t^2-t)\,dt & \int 3\,dt \\ \int (t-1)\,dt & \int (t-2)\,dt \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{t^3}{3} - \frac{t^2}{2} + c_1 & 3t + c_2 \\ \frac{t^2}{2} - t + c_3 & \frac{t^2}{2} - 2t + c_4 \end{bmatrix} = \begin{bmatrix} t^3/3 - t^2/2 & 3t \\ \frac{t^2}{2} - t & \frac{t^2}{2} - 2t \end{bmatrix} + \underbrace{\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}}_{\text{constant of Int in matrix form}}$$

(b) Compute $\int_0^1 M(t)\,dt$.

$$\int_0^1 M(t)\,dt = \begin{bmatrix} \left(\frac{t^3}{3} - \frac{t^2}{2}\right)\Big|_0^1 & 3t\Big|_0^1 \\ \left(\frac{t^2}{2} - t\right)\Big|_0^1 & \left(\frac{t^2}{2} - 2t\right)\Big|_0^1 \end{bmatrix} = \begin{bmatrix} 1/3 - 1/2 - \left(\frac{0^3}{3} - \frac{0^2}{2}\right) \\ 1/2 - 1 - \left(\frac{0}{2}\right. \end{bmatrix}$$

DNF

① $A_{n\times n}$ it has an inverse $\Leftrightarrow \det(A) \neq 0$

② $A^{-1}_{n\times n}$ is the inverse of A $\Leftrightarrow AA^{-1} = I_{n\times n} = A^{-1}A$

③ $I_{n\times n} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix}$

④ $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ab - bc$

⑤ If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(c) For what values of t, $M(t)$ has inverse?

$$\det(M) = \underbrace{(t^2 - t)}_{t(t-1)}(t-2) - 3(t-1) = (t-1)\big[\underbrace{t(t-2) - 3}_{t^2-2t-3}\big] = (t-1)(t+1)(t-3) \neq 0$$

$t \neq 3$ $\quad t \neq \pm 1$

$$M^{-1}(t) = \frac{1}{\det(M)} \begin{bmatrix} t-2 & -3 \\ 1-t & t(t-1) \end{bmatrix} = \begin{bmatrix} \frac{t-2}{(t^2-1)(t-3)} & \frac{-3}{(t^2-1)(t-3)} \\ \frac{1-t}{(t^2-1)(t-3)} & \frac{t(t-1)}{(t^2-1)(t-3)} \end{bmatrix}$$

(d) Find $(M(t))^{-1}$, whenever it makes sense.

Note: In general we will use Gaussian elimination to compute $(M(t))^{-1}$. However, if M is a 2×2 matrix we have that

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

BASIC RULES:

$$\frac{d}{dt}\Big\{A(t) \pm B(t)\Big\} = \frac{d}{dt}\Big\{A(t)\Big\} \pm \frac{d}{dt}\Big\{B(t)\Big\}$$

$$\frac{d}{dt}\Big\{A(t) \cdot B(t)\Big\} = A(t) \cdot B'(t) + A'(t) \cdot B(t)$$

$$\int \Big\{A(t) \pm B(t)\Big\}\ dt = \int A(t)\ dt \pm \int B(t)\ dt$$

$$\int_a^b \Big\{A(t) \pm B(t)\Big\}\ dt = \int_a^b A(t)\ dt \pm \int_a^b B(t)\ dt$$

WARNING: In general, the product of matrices is not commutative. That is, if P and Q are matrices, then PQ may not be QP. Hence, $\frac{d}{dt}\Big\{B(t) \cdot A(t)\Big\}$ may not be $A(t) \cdot B'(t) + A'(t) \cdot B(t)$.

Sec 4.2: First Order Linear System

Has the standard form:

$$\vec{Y}\,' = P(t) \cdot \vec{Y} + \vec{G}(t), \qquad a < t < b$$

where

$$\vec{Y} = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ \cdot \\ \cdot \\ y_n(t) \end{bmatrix}, \qquad P(t) = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ p_{31}(t) & p_{32}(t) & \cdots & p_{3n}(t) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{bmatrix} \qquad \vec{G}(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ g_3(t) \\ \cdot \\ \cdot \\ g_n(t) \end{bmatrix}$$

Note: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{pmatrix}$

Ex1. Write the following system as a first order linear system. Assume $0 < t < \infty$.

$$y_1' = \sin(t) \cdot y_1 + \frac{t}{t^2 - 2t + 8} \cdot y_2 + \ln(t)$$

$$y_2' = (2t + 1) \cdot y_1 + e^{-2t} \cdot y_2 + \cos(t)$$

$\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$ $\quad \vec{y}\,' = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \underbrace{\begin{bmatrix} \sin(t) & \frac{t}{t^2-2t+2} \\ 2t+1 & e^{-2t} \end{bmatrix}}_{P(t)} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \ln(t) \\ \cos(t) \end{pmatrix}$

$\vec{y}(t) + \vec{G}(t)$

$\neq 0 \rightarrow$ non homogeneous system.

Ex2. Write the following system as a first order linear system. Assume $-2 < t < 2$.

Ex2. Write the following system as a first order linear system. Assume $-2 < t < 2$.

$$(t+2)y_1' = 3ty_1 + 5y_2$$

$$(t-2)y_2' = 2y_1 + 4ty_2$$

$$y_1' = \sin(t) \cdot y_1 + \frac{t}{t^2 - 2t + 8} \cdot y_2 + \ln(t)$$

$$y_2' = (2t+1) \cdot y_1 + \mathrm{e}^{-2t} \cdot y_2 + \cos(t)$$

1. SF ✓
2. Find $P(t)$, $\vec{G}(t)$
3. Look at possible discontinuities of $P(t)$ and components of $\vec{G}(t)$
4. Place $t_0 = 1$ in $\mathbb{R}$
5. Conclusion: theorem guarantees the solution of a unique solution over $(0,\infty)$

Ex3. Rewrite the scalar differential equation as a first order linear system:

$$y^{(3)} - t^2 y'' + 3ty' + 5y = e^{-4t}$$

Sol.
Note that this is a third order **scalar** differential equation. Define a column vector $\vec{Y}(t)$ as follows:

$$\vec{Y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \end{bmatrix}$$

Differentiate the column vector $\vec{Y}(t)$ with respect to t:

$$\vec{Y}\,'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{bmatrix} = \begin{bmatrix} y'(t) \\ y''(t) \\ y'''(t) \end{bmatrix}$$

Now use the definition of $\vec{Y}(t)$ and the fact that $y^{(3)} = t^2 y'' - 3ty' - 5y + e^{-4t}$.

$$\vec{y}(t) = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} \qquad y'(t) = \begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ y''' \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ e^{-4t} - 5y_1 - 3ty_2 + t^2 y_3 \end{pmatrix}$$

From the scalar D.E. I solve for y'' in terms of y_1 and y_2

linear system

$$y''' = e^{-4t} \underbrace{-5y_1}_{y} \underbrace{-3ty_2}_{y'} \underbrace{+t^2 y_3}_{y''}$$

$$y' = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -3t & t^2 \end{bmatrix}}_{p(t)} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ e^{-4t} \end{bmatrix}}$$

$\vec{y}(t) + \vec{G}(t) \neq 0$ (non homogeneous)

Definition: The **trace of a matrix** A denoted by $tr[A]$, is defined to be the sum of the elements of the diagonal of A.

Example: Consider A from our previous example,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -6 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then $tr[A] = 0 + 5 + 1 = 6$.

Remember, given a 3×3 matrix $B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$,

determinant of B= det (B) $= a\times$ det $\begin{bmatrix} e & f \\ h & k \end{bmatrix} - b\times$ det $\begin{bmatrix} d & f \\ g & k \end{bmatrix} + c\times$det $\begin{bmatrix} d & e \\ g & h \end{bmatrix}$

Example: Calculate det (A).

Det (A)= $0\times$ det $\begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} - 1\times$ det $\begin{bmatrix} -6 & 0 \\ 0 & 1 \end{bmatrix} + 0\times$ det $\begin{bmatrix} -6 & 5 \\ 0 & 0 \end{bmatrix} = 6$